# MATHEMATICAL METHODS IN THE SOCIAL SCIENCES, 1959

Proceedings of the First Stanford Symposium

Edited by KENNETH J. ARROW
SAMUEL KARLIN
PATRICK SUPPES

1960
STANFORD UNIVERSITY PRESS
STANFORD, CALIFORNIA

STANFORD UNIVERSITY PRESS
STANFORD, CALIFORNIA


LIBRARY OF CONGRESS CATALOG CARD NUMBER: 60-9048
PRINTED IN THE UNITED STATES OF AMERICA

# 15

## A Theory of Stimulus Discrimination Learning

RICHARD C. ATKINSON
University of California, Los Angeles

### 1. Introduction

This paper presents a theory of discrimination learning. For simplicity, the analysis will be restricted to situations in which only one of two stimuli is presented on each trial; the subject's task is to learn to respond appropriately to them. However, the theory can be readily generalized to situations involving more than two stimuli.

There are several recent quantitative theories dealing with this type of problem; in particular, the work of Bush and Mosteller [4], Restle [15, 16, 17], Burke and Estes [3], Green [11], Atkinson [1], and Estes [6, 7]. Each of these theories has certain limitations. The limitations are of two types. In some cases the conceptual framework on which the theory is based seems unrealistic from a psychological viewpoint. In other cases, the theories appear to be contradicted by experimental data or are extremely restrictive in the type of problem to which they are applicable. An analysis of these theories will not be presented (see Restle [18] and Estes [7]) but later some predictions derived from our model and comparable results from other theories will be examined.

The experimental situation involves a series of discrete trials. On each trial one of two stimuli ($S_1$ or $S_2$) is presented. To the presentation of $S_i$, the subject makes one of two responses ($A_1$ or $A_2$); these responses are mutually exclusive and exhaustive. A trial is terminated by a reinforcing event ($E_1$ or $E_2$). If an $E_1$ occurs the $A_1$ response has been reinforced, and if an $E_2$ occurs the $A_2$ response has been reinforced.

Thus the experimenter can present one of the following four combinations on each trial: $(S_1, E_1)$, $(S_1, E_2)$, $(S_2, E_1)$, or $(S_2, E_2)$. The respective probabilities of these four events will be $a, b, c$, and $d$, where $a + b + c + d = 1$. More general schedules of reinforcing events and stimulus events can be investigated but the above schedule encompasses most of the experimental research on discrimination learning. The traditional type of discrimination task is described when $a = d = 1/2$ and $b = c = 0$; the subject must learn to

This research was supported by the Office of Naval Research under Contract Nonr 233(58).

make the $A_1$ response to the presentation of $S_1$ and to make the $A_2$ response to the presentation of $S_2$. Another type of discrimination problem only recently investigated is specified when $a, b, c, d \neq 0$ [2], [6], [8], [9], [14].

Three additional parameters are frequently used to avoid cumbrous notation: $\pi_1 = a/(a+b)$, $\pi_2 = c/(c+d)$, and $\beta = a + b$. The parameter $\pi_1$ is the probability of an $E_1$ event given $S_1$, the parameter $\pi_2$ is the probability of an $E_1$ event given $S_2$, and $\beta$ is the probability of an $S_1$ presentation.

The aim of the present model is to account for the following factors in discrimination learning:

(i) The effect of stimulus dimensions; specifically, the variable of stimulus similarity or differentiability;

(ii) The effect of reinforcement schedules; the influence of $\pi_1$ and $\pi_2$;

(iii) The effect of stimulus schedules; that is, the influence of variations in $\beta$;

(iv) Previous experience on other discrimination tasks.

Other variables influencing discrimination learning have been experimentally investigated, but for the moment we shall be more than happy to limit our attention to the above factors.

The dependent variable of major interest is the expected probability on trial $n$ of an $A_i$ response to the presentation of an $S_j$ stimulus. This probability will be denoted as $p_n(A_i \mid S_j)$.

The model is similar in some respects to the theories mentioned above. All of these theories conceptually represent the $S_i$ stimulus as a collection of component parts; the hypothetical components are called stimulus elements or cues. Thus, the stimuli $S_1$ and $S_2$ are represented by two sets of elements, and similarity between the stimuli is defined with respect to the number of elements the two sets have in common. Some theorists have proposed that individual elements are conditioned to responses; these theories will be referred to as component models. Others, notably Estes [7], have proposed that patterns of stimulus elements are conditioned to responses.

The problem encountered by both types of theories is illustrated by the observation that subjects can learn to discriminate with perfect accuracy between stimuli having so many features in common that initially there is marked generalization among them. To account for this result the component theories have had to postulate some mechanism by which the common stimulus elements gradually become ineffective or, to use Restle's terminology, become "adapted and rendered nonfunctional during learning." The mechanisms that have been proposed lack psychological rationale and tend to be applicable to extremely restricted schedules of reinforcement. The pattern model is not subject to these objections and has no difficulty accounting for the above observation. Unhappily, however, it implies that the rate of discrimination learning is independent of the number of common stimulus elements. This prediction seems contrary to most experimental evidence.

In our model, both conditioning concepts are employed. Further, a mechanism is postulated that integrates the two types of conditioning. The control of the integrating mechanism is governed by the reinforcing schedule and stimulus similarity of the particular discrimination task.

# THEORY

## 2. Stimulus Representation

Stimuli impinging on the organism are to be represented conceptually in terms of a set $S^*$ of *stimulus elements*. The presentation of a particular stimulus leads to the *activation* of a unique subset of $S^*$ with probability 1. Further, as will be indicated later, the response elicited by the stimulus is a function of the activated stimulus elements. These stimulus elements are theoretical constructs to which we will assign certain properties. They are not the receptor neurons of neurophysiology but a schematic representation of the physical stimulus, having certain simple and uniform properties.

For our problem, the total stimulus situation associated with the presentation of $S_1$ and the total stimulus situation associated with the presentation of $S_2$ are represented by two subsets of stimulus elements $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, respectively; the presentation of $S_i$ $(i = 1, 2)$ leads to the activation of set $\boldsymbol{S}_i$. For different pairs of physical stimuli the sets $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ may be of different sizes and have different relationships. That is, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ may be disjoint, partially overlapping, or one may be a subset of the other.

A set $\boldsymbol{S}_c$ is defined that represents those stimulus elements common to $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ (i.e., $\boldsymbol{S}_c = \boldsymbol{S}_1 \cap \boldsymbol{S}_2$). Typically, two sources of common elements are identified in a discrimination problem. One subset of common elements may be associated with the actual stimuli to be discriminated; the other may be associated with background stimuli such as characteristics of the experimental chamber, sounds from the apparatus, proprioceptive stimulation, and so forth.

Let $n(\boldsymbol{S}_1)$, $n(\boldsymbol{S}_2)$, and $n(\boldsymbol{S}_c)$ represent the number of elements in $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, and $\boldsymbol{S}_c$, respectively. Then

$$(1) \qquad \omega_1 = \frac{n(\boldsymbol{S}_1) - n(\boldsymbol{S}_c)}{n(\boldsymbol{S}_1)}, \qquad \omega_2 = \frac{n(\boldsymbol{S}_2) - n(\boldsymbol{S}_c)}{n(\boldsymbol{S}_2)} .$$

The quantities $\omega_1$ and $\omega_2$ can be used to define indexes of similarity between the stimuli $S_1$ and $S_2$ similar to those introduced by Bush and Mosteller [4]. Roughly speaking, the more dissimilar the stimuli the closer $\omega_1$ and $\omega_2$ are to unity; as similarity increases $\omega_1$ and $\omega_2$ approach zero.

For many experimental situations the stimuli $S_1$ and $S_2$ are essentially comparable (see, for example, [2]) and, consequently, it is natural to assume that $\omega_1 = \omega_2 = \omega$; hence, when the subscript on $\omega$ is omitted the reader should assume that $\omega_1 = \omega_2$.

One additional restriction is imposed; namely, $0 < \omega_1 \leqq 1$ and $0 \leqq \omega_2 \leqq 1$. That is, the case where $\omega_1 = \omega_2 = 0$ will not be considered. This would describe a situation in which the same stimulus (represented by $\boldsymbol{S}_c$) is presented on all trials of the experiment. The case is considered in detail by Suppes and Atkinson [19] for models of the type to be presented in this paper.

### 3. Conditioning States

With regard to conditioning, we distinguish between individual stimulus elements and *stimulus patterns*. A stimulus pattern refers to a subset of stimulus elements all of which are activated simultaneously; on a given trial, only one stimulus pattern can be activated. Two conditioning relations are defined: (i) on trial $n$, each stimulus element is conditioned to either $A_1$ or $A_2$; (ii) on trial $n$, each stimulus pattern is conditioned to either $A_1$ or $A_2$.

To clarify, consider a set $S^*$ with only three stimulus elements $s_1, s_2$, and $s_3$. There are seven possible stimulus patterns: $(s_1), (s_2), (s_3), (s_1, s_2), (s_1, s_3)$, $(s_2, s_3)$, and $(s_1, s_2, s_3)$. If the stimulus impinging on the subject is such that the elements $s_1, s_2$, and $s_3$ are activated simultaneously, then the only *pattern* activated is $(s_1, s_2, s_3)$; patterns $(s_1, s_3)$ and $(s_2, s_3)$ are not activated. Under the most general experimental conditions we might assume that any set of conditioning relations is possible; as an example, to be referred to again, the following states might hold on trial $n$ for individual elements and for the stimulus patterns:

$$
\begin{aligned}
&s_1 - A_2,\ s_2 - A_1,\ s_3 - A_1,\ (s_1) - A_1\ ,\\
&(s_2) - A_2,\ (s_3) - A_2,\ (s_1, s_2) - A_1\ ,\\
&(s_1, s_3) - A_2,\ (s_2, s_3) - A_1,\ (s_1, s_2, s_3) - A_2\ .
\end{aligned}
\tag{2}
$$

In the next two sections we consider how these conditioning relations determine responses and how they are established.

### 4. Response Probability and Perceptual States

The response made by a subject on a given trial depends on the conditioning states of the activated stimulus elements. However, the response can be determined either by the stimulus pattern or by the individual stimulus elements. Specifically,

($P$) The subject can respond in terms of the activated stimulus pattern and make the $A_i$ response to which the activated stimulus pattern is conditioned;

($C$) The subject can respond to the component stimulus elements such that the probability of an $A_i$ response is the proportion of activated stimulus elements conditioned to $A_i$.

If the response is determined on trial $n$ by the stimulus pattern, then we will say that the subject is in state $P$ on trial $n$; if the response is determined by the component stimulus elements, then the subject is in state $C$. These two states will be referred to collectively as *perceptual states*.

In (2), if the stimulus elements $s_1$, $s_2$, and $s_3$ are activated simultaneously, then (i) an $A_2$ will be elicited if the subject is in perceptual state $P$, and (ii) an $A_1$ will be elicited with probability 2/3 if the subject is in state $C$. However, if only $s_1$ is activated, then (i) an $A_1$ will be made if the subject is in state $P$, and (ii) an $A_2$ will occur if the subject is in state $C$.

Whether a subject is in the $P$ or $C$ state at the start of trial $n$ depends on his reinforcement history. If the sequence of $A_i$ responses made by the subject has been consistently reinforced when the subject was in the $P$ state, then he will remain in state $P$; if the $A_i$ responses have been consistently reinforced when the subject was in the $C$ state, then he will remain in state $C$. If partial reinforcement has occurred when the subject was in either $P$ or $C$, then in the future the subject will be in state $P$ or $C$ with some probability other than 0 or 1.

## 5. Conditioning Process

To be explicit, in discussing conditioning we introduce the notion of a random variable. The formal application of random variables could have been made in describing other features of the model, but until now it would not have facilitated the presentation. We remind the reader that a random variable is a (measurable) function defined on a sample space, and the over-all probability measure on the sample space induces a probability distribution on the values of the random variable.

The sample space $X$ for a given experiment is the set of all possible outcomes $x$ of the experiment. It is suggestive to think of $x$ as the sequence of stimulus, response, and reinforcing events for a particular subject in the experiment, but it is important to remember that when in defining the random variable we speak of subject $x$, we mean the subject's protocol in a particular *realization* of the experiment. The same subject would undoubtedly produce a different experimental outcome $x$ in another realization of the experiment at another time.

The random variable $F_n(x)$ describes changes in conditioning and perceptual states as a function of reinforcement schedules. Specifically, the conditioning random variable is defined in terms of the following statements:

(i) All activated stimulus elements and the activated stimulus pattern are conditioned to the reinforced response. That is, if $E_j$ occurs on trial $n$, then the stimulus elements and the stimulus pattern activated on the trial are conditioned to $A_j$.

(i′) No change occurs in the conditioning state of stimulus elements or stimulus patterns on trial $n$.

(ii) If the subject is in perceptual state $P$ and the $A_j$ response made is reinforced, then the subject will remain in state $P$; however, if the alternative response is reinforced, then the subject will change to state $C$. Similarly, for the $C$ state, if the response emitted is reinforced the subject remains in $C$, but he switches to $P$ if the $A_j$ response was not reinforced.

(ii′) No change occurs in the perceptual state on trial $n$.

Then

$$F_n(x) = \begin{cases} 3 & \text{if (i) and (ii) apply ,} \\ 2 & \text{if (i) and (ii}') \text{ apply ,} \\ 1 & \text{if (i}') \text{ and (ii) apply ,} \\ 0 & \text{if (i}') \text{ and (ii}') \text{ apply .} \end{cases} \tag{3}$$

Only three cases will be considered in this paper.

Case I:

$$P(F_n = 3) = \theta, \qquad P(F_n = 2) = P(F_n = 1) = 0, \qquad P(F_n = 0) = (1 - \theta) \qquad (0 < \theta \leqq 1)\,. \tag{4}$$

Case II:

$$P(F_n = 3) = \alpha_1\alpha_2, \qquad P(F_n = 2) = (1 - \alpha_1)\alpha_2, \qquad P(F_n = 1) = \alpha_1(1 - \alpha_2), \qquad P(F_n = 0) = (1 - \alpha_1)(1 - \alpha_2) \qquad (0 < \alpha_1,\ \alpha_2 \leqq 1)\,. \tag{5}$$

Case III:

$$P(F_n = 3) = \gamma_3, \qquad P(F_n = 2) = \gamma_2, \qquad P(F_n = 1) = \gamma_1, \qquad P(F_n = 0) = \gamma_0\,. \tag{6}$$

## 6. Subject States

At the start of trial $n$, it is assumed that a subject can be described by an ordered four-tuple in which

(i) The first member of the tuple is either $P$ or $C$ and indicates the perceptual state on trial $n$.

(ii) The second member is either $A_1$ or $A_2$. If it is $A_i$ $(i = 1, 2)$, then the individual stimulus elements in the subset $S_1 \sim S_c$ are conditioned to $A_i$ and the stimulus pattern associated with the entire subset $S_1$ is conditioned to $A_i$.

(iii) The third member is either $A_1$ or $A_2$ and indicates whether the individual elements in $S_c$ are conditioned to $A_1$ or $A_2$.

(iv) The fourth member is either $A_1$ or $A_2$. If it is $A_i$ $(i = 1, 2)$, then the individual stimulus elements in the subset $S_2 \sim S_c$ are conditioned to $A_i$ and the stimulus pattern associated with the entire subset $S_2$ is conditioned to $A_i$.

Thus, following the rules for response probability, we find that if the subject is in state $(C, A_1, A_2, A_2)$, then the probability of an $A_1$ response is $\omega_1$ in the presence of $S_1$ and 0 in the presence of $S_2$. As a second example, for the state $(P, A_1, A_2, A_1)$ the probability of an $A_1$ response is 1 in the presence of both $S_1$ and $S_2$.

These four-tuples will be referred to as *subject states* and assigned identifying numbers as follows:

| | | | |
|---|---|---|---|
| 1. | $(P, A_1, A_1, A_1)$ | 9. | $(C, A_1, A_1, A_1)$ |
| 2. | $(P, A_1, A_1, A_2)$ | 10. | $(C, A_1, A_1, A_2)$ |
| 3. | $(P, A_1, A_2, A_1)$ | 11. | $(C, A_1, A_2, A_1)$ |
| 4. | $(P, A_1, A_2, A_2)$ | 12. | $(C, A_1, A_2, A_2)$ |
| 5. | $(P, A_2, A_1, A_1)$ | 13. | $(C, A_2, A_1, A_1)$ |
| 6. | $(P, A_2, A_1, A_2)$ | 14. | $(C, A_2, A_1, A_2)$ |
| 7. | $(P, A_2, A_2, A_1)$ | 15. | $(C, A_2, A_2, A_1)$ |
| 8. | $(P, A_2, A_2, A_2)$ | 16. | $(C, A_2, A_2, A_2)$ |

The subject state description permits response distributions to be specified in a fairly simple fashion. One reason for being able to restrict the number of states to sixteen is that, by assumption, if conditioning has occurred on a trial (i.e., if $F_n(x) = 3, 2$) in which $S_j$ ($j = 1, 2$) was presented, then on all subsequent trials the stimulus pattern associated with $S_j$ and the individual stimulus elements in the subset $S_j \sim S_c$ will both be conditioned to the same response. Thus it is not necessary to keep track separately of the stimulus pattern for $S_j$ and the individual elements in the subset $S_j \sim S_c$, as indicated in the description of the second and fourth components of a subject state. The only complication is with regard to initial conditions, where we might wish to admit the possibility that the pattern associated with $S_j$ and the elements in the subset $S_j \sim S_c$ are conditioned to different responses. This is a limitation but it is offset by the resulting simplicity of the model.

### 7. Mathematical Formulation

From the assumptions presented in the preceding sections, it can be shown that the sequence of random variables that take the subject states as values is a Markov chain.[1] This means, among other things, that a transition matrix $P = [p_{ij}]$ may be defined, where $p_{ij}$ is the conditional probability of being in subject state $j$ on trial $n + 1$, given state $i$ on trial $n$. Information about subject states on trials preceding $n$ in no way affects $p_{ij}$ on trial $n$; further, $p_{ij}$ is independent of $n$, that is, constant over trials. The learning process is completely characterized by these transition probabilities and the initial probability distribution on the subject states.

We now use the axioms of the preceding section to derive the transition matrix. In making such a derivation it is convenient to represent the various possible occurrences on a trial by a tree. Each set of branches emanating from a point represents a mutually exclusive and exhaustive set of possibilities. Two of the sixteen trees are presented in Figures 1 and 2 to illustrate the procedure.

Each path on a tree from a beginning point to a terminal point represents a possible change in subject state on a given trial. The probability of each path is obtained by multiplication of conditional probabilities. Thus, for Figure 1 the probability of the top path is simply $\gamma_3 a$. The probability of a transition from one state to another is obtained by summing over appropriate branches. For example, of the sixteen paths in Figure 1, two of them lead from state 4 to state 2 and, therefore, $p_{4,2} = \gamma_3 a + \gamma_2 a$. As a second example, in Figure 2 only one path leads from state 12 to state 2 and, consequently, $p_{12,2} = \gamma_3(1 - \omega_1)a$. The complete set of transition probabilities for Case III is given in the Appendix.

Let $u_i(n)$ be the probability of being in subject state $i$ at the start of trial $n$, where $n = 1, 2, \cdots$. Define the row matrix

$$U(n) = [u_1(n), u_2(n), \cdots, u_{16}(n)] . \tag{7}$$

Then $U(n) = U(n-1) \cdot P$, and, in general,

[1] For more complex stimulus and reinforcement schedules or for conditioning assumptions other than those proposed in Cases I–III, the particular sequence of Subject states may not be a Markov chain.

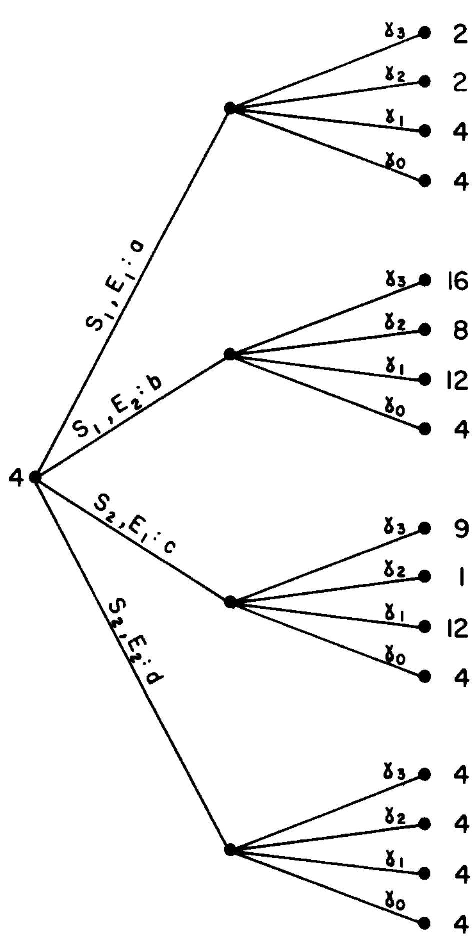


Figure 1. A tree listing possible subject states on trial $n+1$, given state 4 on trial $n$.

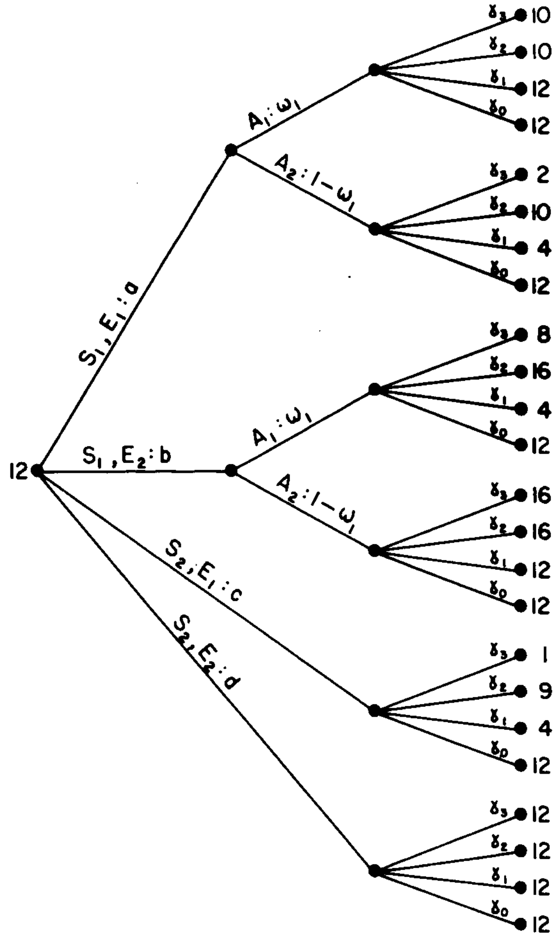


Figure 2. A tree listing possible states on trial $n+1$, given state 12 on trial $n$.

(8) $$U(n) = U(1) \cdot P^{n-1} .$$

Further, define $p_{ij}^{(n)}$ as the probability of being in state $j$ at trial $r+n$, given that at trial $r$ we were in state $i$. Moreover, if the appropriate limit exists and is independent of $i$, let

(9) $$u_j = \lim_{n\to\infty} p_{ij}^{(n)} .$$

The limiting quantities $u_j$ exist for any finite-state Markov chain that is irreducible and aperiodic. A Markov chain is irreducible if there is no closed proper subset of states, that is, no proper subset of states such that once within this set the probability of leaving it is 0. A Markov chain is aperiodic if there is no fixed period for return to any state.

Experimentally, it is impossible to identify individual states of the process on a given trial. That is, the experimenter knows which stimulus ($S_1$ or $S_2$), response ($A_1$ or $A_2$), and reinforcing event ($E_1$ or $E_2$) occurred on the trial, but this information is not sufficient to identify the subject state. For example, if $S_1$ is presented and $A_1$ occurs, we cannot establish unequivocally which of the sixteen states the subject was in when the $A_1$ occurred. In fact, for this particular combination, any one of the following ten states would have been possible: 1, 2, 3, 4, 9, 10, 11, 12, 13, or 14. Obviously, this confounding is due to the fact that the perceptual states are not directly observable.

Since trial descriptions and subject states cannot be placed in one-to-one correspondence, it is necessary (for an experimental evaluation of the theory) to define probabilities of events that are observable. Consequently, the following quantities are of particular interest: $p_n(A_1 \mid S_1)$, the conditional probability on trial $n$ of an $A_1$ response given an $S_1$ stimulus presentation; and $p_n(A_1 \mid S_2)$, the conditional probability on trial $n$ of an $A_1$ response given an $S_2$ stimulus presentation. By inspection of the theoretical states it follows that

(10) $$\begin{aligned} p_n(A_1 \mid S_1) = {} & u_1(n) + u_2(n) + u_3(n) + u_4(n) + u_9(n) + u_{10}(n) \\ & + \omega_1[u_{11}(n) + u_{12}(n)] + (1-\omega_1)[u_{13}(n) + u_{14}(n)] \end{aligned}$$

and

(11) $$\begin{aligned} p_n(A_1 \mid S_2) = {} & u_1(n) + u_3(n) + u_5(n) + u_7(n) + u_9(n) + u_{13}(n) \\ & + \omega_2[u_{11}(n) + u_{15}(n)] + (1-\omega_2)[u_{10}(n) + u_{14}(n)] . \end{aligned}$$

Also, for analytical purposes, the probabilities at the start of trial $n$ of perceptual state $P$ will be useful:

(12) $$p_n(P) = u_1(n) + u_2(n) + \cdots + u_8(n) .$$

## ANALYSIS OF THE MODEL

### 8. Empirical Implications of Case I

In this case changes in the conditioning and perceptual states are completely dependent on each other, as indicated by (4). If, on trial $n$, a change in the conditioning state is possible, then a change in the perceptual state also can occur; if a change is not possible in the conditioning state, then no change can occur in the perceptual state.

An inspection of the transition matrix for this case leads to two immediate conclusions: (i) the subject states 3, 6, 11, and 14 are transient; (ii) the asymptotic probability of any subject state is independent of $\theta$. This follows from the observation that the main diagonal of $P$ for this case has terms of the form $(1-\theta)+\theta\delta$, while all other non-zero terms are of the form $\theta\delta'$.

*Asymptotic Predictions*

We now consider some general aspects of asymptotic behavior that are of experimental interest. If $0<\omega_1<1$ and $0 \leqq \omega_2 < 1$, then a single closed set of states exists,[2] namely $C_1 = \{2, 4, 5, 7, 9, 16\}$; $C_1$ forms an irreducible aperiodic Markov chain, and it can be shown that

$$(13)\qquad \begin{aligned} u_2 &= ad(a+b)/\Delta\,, & u_7 &= bc(a+b)/\Delta, \\ u_4 &= ad(c+d)/\Delta\,, & u_9 &= ac/\Delta, \\ u_5 &= bc(c+d)/\Delta\,, & u_{16} &= bd/\Delta\,, \end{aligned}$$

where $\Delta = (a+b)(c+d)$.

Further, by (10)–(12) we have

$$(14)\qquad p_\infty(A_1 \mid S_1) = u_2 + u_4 + u_9 = \pi_1, \qquad p_\infty(A_1 \mid S_2) = u_5 + u_7 + u_9 = \pi_2,$$
$$p_\infty(P) = u_2 + u_4 + u_5 + u_7 = \frac{ad+bc}{\Delta}\,.$$

The only other admissible condition not considered above is when $\omega_1 = \omega_2 = 1$. In this event, two closed sets of states exist, namely $C_1$ and $C_2 = \{1, 8, 10, 12, 13, 15\}$. If the subject is not in $C_1$ or $C_2$ at the start of the experiment, then (i) the subject will be absorbed into $C_1$ if he was initially in state 11 or 14, and (ii) the subject will be absorbed into $C_2$ if he was initially in state 3 or 6. Thus when $\omega_1 = \omega_2 = 1$ the probabilities of absorption into $C_1$ and $C_2$ are, respectively:

$$(15)\qquad u_2(1) + u_4(1) + u_5(1) + u_7(1) + u_9(1) + u_{11}(1) + u_{14}(1) + u_{16}(1)$$

and

$$(16)\qquad u_1(1) + u_3(1) + u_6(1) + u_8(1) + u_{10}(1) + u_{12}(1) + u_{13}(1) + u_{15}(1)\,.$$

$C_2$ also forms an irreducible aperiodic Markov chain, and if the subject is absorbed into the $C_2$ set, then

$$(17)\qquad \begin{aligned} u_1 &= ad/\Delta, & u_{12} &= ad(c+d)/\Delta\,, \\ u_8 &= bd/\Delta, & u_{13} &= bc(c+d)/\Delta\,, \\ u_{10} &= ad(a+b)/\Delta, & u_{15} &= bc(a+b)/\Delta\,. \end{aligned}$$

Further, when absorbed into $C_2$,

[2] A set $C$ of states is closed if no state outside $C$ can be reached from any state in $C$.

(18) $$p_\infty(A_1 \mid S_1) = u_1 + u_{10} + u_{12} = \pi_1, \qquad p_\infty(A_1 \mid S_2) = u_1 + u_{13} + u_{15} = \pi_2 ,$$
$$p_\infty(P) = u_1 + u_8 = \frac{ac + bd}{\Delta} .$$

Thus for Case I, $p_\infty(A_i \mid S_j)$ is independent of $\theta, \omega_1, \omega_2$, and the initial probability distribution $U(1)$, as indicated by (14) and (18). The same conclusion holds for $p_\infty(P)$ except for the case where $\omega_1 = \omega_2 = 1$; here,

(19) $$p_\infty(P) = \begin{cases} (ab + bc)/\Delta, & \text{with probability indicated by (15)} , \\ (ac + bd)/\Delta, & \text{with probability indicated by (16)} . \end{cases}$$

Thus, when $\omega_1 = \omega_2 = 1$, the asymptotic probability $p_\infty(P)$ depends on $U(1)$.

The asymptotic results for $p_\infty(A_i \mid S_j)$ correspond to the "matching law" described by W. K. Estes for a linear learning model [5]. That is, asymptotically the probability of a response in the presence of a given stimulus is simply the probability of reinforcing that response when the stimulus is presented. That is, the probability of an $E_1$ event given $S_1$ is $\pi_1$, which is $p_\infty(A_1 \mid S_1)$; similarly, the probability of an $E_1$ given $S_2$ is $\pi_2$, which is $p_\infty(A_1 \mid S_2)$. Experimental evidence dealing with this result will be presented later.

*Pre-asymptotic Predictions*

Certain pre-asymptotic properties of the model for a classical discrimination task will be investigated, that is, for a situation in which $a = d = 1/2$ and $b = c = 0$. For this problem $A_1$ is correct when $S_1$ is presented and $A_2$ is correct when $S_2$ is presented.

Under these conditions the process is absorbed in a closed set consisting of states 2 and 4. Consequently,

(20) $$p_\infty(A_1 \mid S_1) = p_\infty(A_2 \mid S_2) = p_\infty(P) = 1 .$$

To simplify the analysis, assume that $\omega_1 = \omega_2 = \omega$ and $u_i(1) = 1/16$ ($i = 1, 2, \cdots, 16$). Both of these assumptions are reasonable for many experimental applications. The two stimuli are taken to be comparable by the first condition and no initial response bias is posited by the second condition.

Given these restrictions, it is obvious that $p_n(A_1 \mid S_1) = p_n(A_2 \mid S_2)$ for all $n$. Further, it can be shown that

(21) $$\begin{aligned} u_3(n) &= u_5(n) = u_6(n) = u_7(n) = u_{11}(n) \\ &= u_{13}(n) = u_{14}(n) = u_{15}(n) = \frac{1}{16}(1 - \theta)^{n-1} , \\ u_1(n) &= u_8(n) = u_9(n) = u_{16}(n) = \frac{1}{4}(1 - \theta/2)^{n-1} - \frac{3}{16}(1 - \theta)^{n-1}, \\ u_{10}(n) &= u_{12}(n) = \frac{1}{4\omega}\{[1 - \theta(1 - \omega)/2]^{n-1} - [1 - \theta/2]^{n-1}\} + \frac{1}{16}(1 - \theta)^{n-1}, \\ u_2(n) &= u_4(n) . \end{aligned}$$

By (21) and the fact that $\sum u_i(n) = 1$, it follows that

$$(22) \qquad p_n(A_1 \mid S_1) = 1 - \frac{1-\omega}{4\omega}[1 - \theta(1-\omega)/2]^{n-1} - \frac{3\omega - 1}{4\omega}[1 - \theta/2]^{n-1} .$$

Equation (22) also describes the probability of a correct response, defined here as

$$p_n(C) = p_n(A_1 \mid S_1) \cdot P_n(S_1) + p_n(A_2 \mid S_2) \cdot P_n(S_2) = p_n(A_1 \mid S_1) .$$

An inspection of (22) indicates that the rate of approach to the asymptote is a function of both $\theta$ and $\omega$. To investigate the effect of $\omega$ on conditioning, let $\theta$ remain fixed and consider two discrimination problems, one described by $\omega'$ and the other by $\omega$, where $\omega' > \omega$. Further, for fixed $\theta$, define $p'_n(C)$ and $p_n(C)$ in terms of $\omega'$ and $\omega$, respectively.

By inspection of (22) it is clear that on early trials $p'_n(C)$ will be greater than $p_n(C)$. However, at some trial a crossover occurs, and from this point on $p_n(C)$ is greater than $p'_n(C)$ as they both approach unity. Further, for a fixed value of $\omega$, the crossover point occurs later in the series of trials as $\omega'$ increases. The prediction of an increase in the number of errors on early trials with an increase in similarity of the discriminanda is reasonably well-established experimentally. With regard to the crossover effect we know of no conclusive experimental evidence; however, some recent data of LaBerge and Smith [13] suggest that such a phenomenon may occur.

The proof of the above result depends on a rather surprising observation. Define $E_N$ as the expected number of incorrect responses made by a subject over the first $N$ trials of the experiment. Then

$$E_N = \sum_{n=1}^{N} [1 - p_n(C)] = \frac{3}{2\theta} - \frac{1}{2\omega\theta}[1 - \theta(1-\omega)/2]^N - \frac{3\omega - 1}{2\omega\theta}[1 - \theta/2]^N .$$

As the number of trials becomes large,

$$E_N \xrightarrow{N} \frac{3}{2\theta} .$$

Thus we have another interesting experimental prediction for Case I. The number of errors made in the "perfect" learning of a classical discrimination problem is independent of the similarity of the discriminanda.

## 9. Empirical Implications of Case II

In this case, changes in the perceptual and conditioning states are statistically independent, as indicated by (5). On any trial, a change in the perceptual state may occur with probability $\alpha_1$, while a change in the conditioning state may occur with probability $\alpha_2$. This modification (as compared with Case I, in which changes in the perceptual and conditioning states are completely dependent) leads to predictions which, for some parameter values, are markedly different from those of Case I.

In analyzing Case II some experimental data will be compared with theoretical predictions. However, before inspecting particular experiments, there are several general results that are important in applying the theory:

(i) As in Case I, subject states 3, 6, 11, and 14 are transient.

(ii) In general, $u_i$ and $p_\infty(A_1 | S_i)$ depend on $\alpha_1, \alpha_2, \omega_1$, and $\omega_2$. However, as $\alpha_1$ and $\alpha_2$ approach 1, the stationary process is described by (14)–(18) and, therefore, by $p_\infty(A_1 | S_1) = \pi_1$ and $p_\infty(A_1 | S_2) = \pi_2$.

(iii) We have restricted $0 < \alpha_i \leqq 1$. However, results relevant to other theories of discrimination obtain when $\alpha_1 = 0$ and $0 < \alpha_2 \leqq 1$. Under these conditions, no change can occur in the perceptual state of the subject. If initially he is in a $P$-state, then he will remain in a $P$-state; the same holds for $C$-states. Consequently, two closed sets are formed, one consisting of states 1 through 8 and the other of states 9 through 16. For the former closed set,

$$\begin{aligned} u_1 &= ac/\Delta , & u_5 &= bc(c+d)/\Delta , \\ u_2 &= ad(a+b)/\Delta , & u_6 &= 0, \\ u_3 &= 0, & u_7 &= bc(a+b)/\Delta , \\ u_4 &= ad(c+d)/\Delta , & u_8 &= bd/\Delta . \end{aligned} \tag{23}$$

If the process is absorbed in this set, by (10) and (11)

$$\begin{aligned} p_\infty(A_1 | S_1) &= u_1 + u_2 + u_3 + u_4 = \pi_1 , \\ p_\infty(A_1 | S_2) &= u_1 + u_3 + u_5 + u_7 = \pi_2 . \end{aligned} \tag{24}$$

The $u_i$'s for the second closed set, consisting of states 9 through 16, are identical to those presented in (23), i.e., $u_i = u_{i+8}$. However, for these states $p_\infty(A_1 | S_i)$ is defined by (10) and (11) as

$$\begin{aligned} p_\infty(A_1 | S_1) &= u_9 + u_{10} + \omega_1[u_{11} + u_{12}] + (1-\omega_1)[u_{13} + u_{14}] \\ &= \pi_1\omega_1 + (1-\omega_1)[\beta\pi_1 + (1-\beta)\pi_2] , \\ p_\infty(A_1 | S_2) &= u_9 + u_{13} + \omega_2[u_{12} + u_{15}] + (1-\omega_2)[u_{10} + u_{14}] \\ &= \pi_2\omega_2 + (1-\omega_2)[\beta\pi_1 + (1-\beta)\pi_2] . \end{aligned} \tag{25}$$

Thus, when $\alpha_1 = 0$, asymptotic behavior is described by (24) or (25), depending on the initial state of the subject. It is interesting to note that the results of (25) are identical to the asymptotic predictions made by Burke and Estes [3]. In contrast, the predictions embodied in (24) are identical to the predictions generated by Estes' pattern model [7].

(iv) Another general result of interest holds for the case in which $\omega_1 = \omega_2 = 1$. Here, independent of the value of $\beta$, we have $p_\infty(A_1 | S_1) = \pi_1$ and $p_\infty(A_1 | S_2) = \pi_2$.

*Classical Discrimination Learning*

When $\pi_1 = 1$, $\pi_2 = 0$, and $\beta \neq 0, 1$, it is clear, by inspection of the transition matrix, that states 2 and 4 form the only closed set and, further, that $u_2 = a$ and $u_4 = d$. Consequently,

$$p_n(A_1 | S_1) \xrightarrow{n} 1, \qquad p_n(A_2 | S_2) \xrightarrow{n} 1 .$$

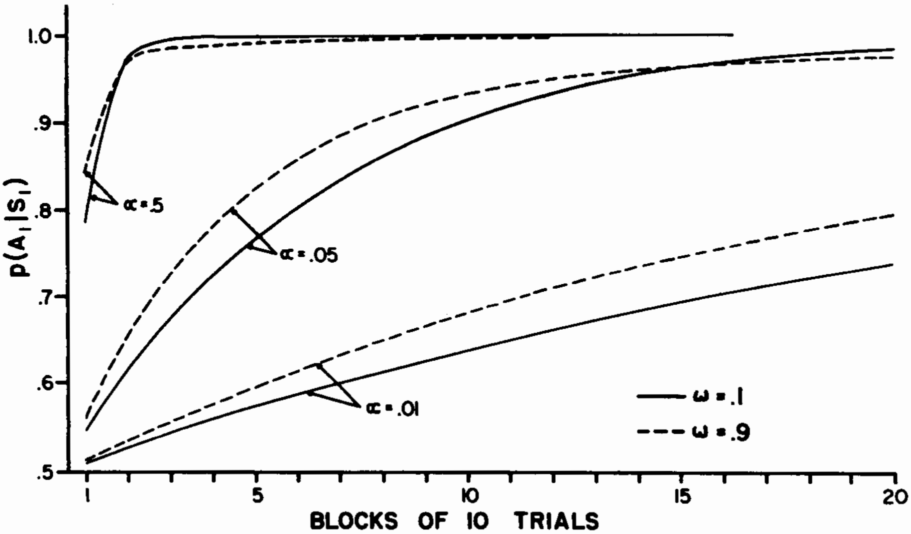


Figure 3. Theoretical functions for $p_n(A_1 \mid S_1)$ averaged over 10-trial blocks; $a = d = 1/2$ and $u_i(1) = 1/16$.

Computational results are presented in Figure 3 for $a = d = 1/2$, $u_i(1) = 1/16$, $\alpha_1 = \alpha_2 = \alpha$, and $\omega_1 = \omega_2 = \omega$. Only $p_n(A_1 \mid S_1)$ is plotted, since it is equal to $p_n(A_2 \mid S_2)$ under these conditions. Clearly, the larger the value of $\alpha$ the more rapidly the subject approaches perfect responding. Also, as was true for Case I, the proportion of correct responses on early trials increases with increasing values of $\omega$; however, at some trial a crossover occurs and from this point on the larger the value of $\omega$ the greater the number of errors.

Similar computations are provided in Figure 4 for $\omega_1 > 0$ and $\omega_2 = 0$. This would describe a situation in which the presentation of the stimulus $S_2$ is theoretically represented by a set of stimulus elements that is a proper subset of $S_1$. For example, experimentally $S_1$ might be the onset of a tone and a light while $S_2$ is the onset of just the tone. As can be seen from the figure, in mastering the problem the subject makes more errors on $S_2$ trials than on $S_1$ trials. Stated differently, the subject learns to make the $A_1$ response to $S_1$ more quickly than the $A_2$ response to $S_2$.

An experiment was conducted to test this specific prediction, employing a procedure and apparatus described elsewhere [2]. Facing the subject was an array of ten lights; the onset of all ten lights designated an $S_1$ trial and the onset of a specific subset of five of these lights designated an $S_2$ trial. (The five lights that made up the $S_2$ were randomly selected for each subject.) On every trial the subject made either an $A_1$ or an $A_2$ response; this was followed by a signal that told him which response was correct. Sixty subjects were run for eighty trials; $a = d = 1/2$, with the restriction that over the eighty trials there were exactly forty $S_1$'s and forty $S_2$'s.

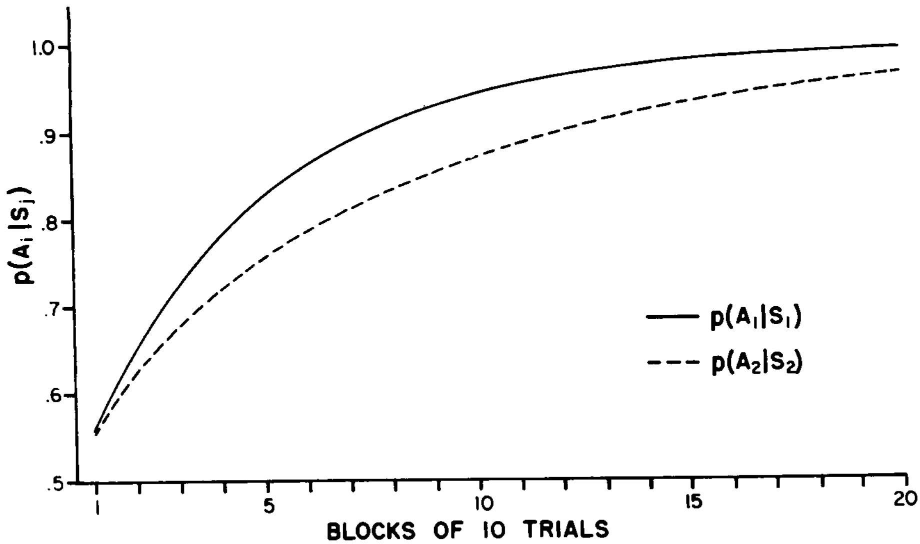


Figure 4(a). Theoretical functions for $p_n(A_i \mid S_i)$ averaged over 10-trial blocks; $a = d = \frac{1}{2}$, $u_i(1) = \frac{1}{16}$, $\alpha_1 = \alpha_2 = .05$, and $\omega_1 = .9$, $\omega_2 = 0$.

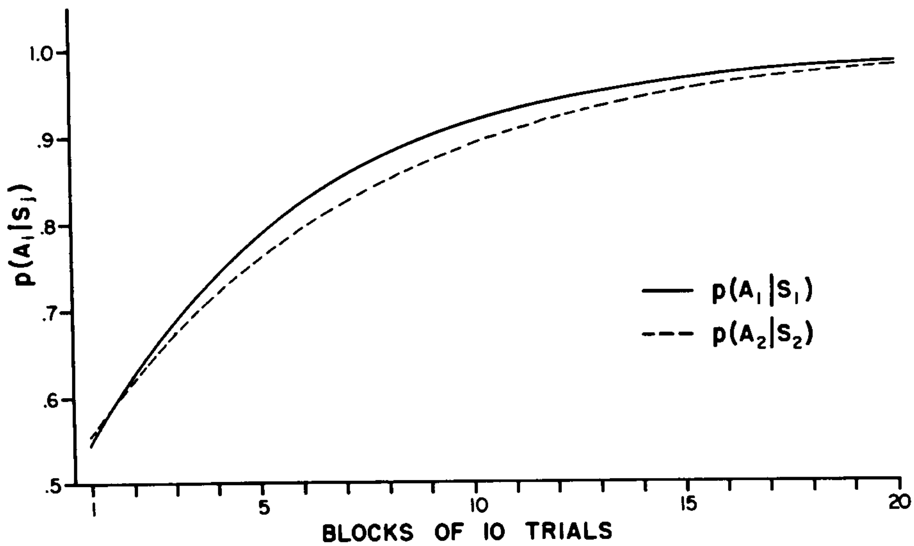


Figure 4(b). Theoretical functions for $p_n(A_i \mid S_i)$ averaged over 10-trial blocks; $a = d = \frac{1}{2}$, $u_i(1) = \frac{1}{16}$, $\alpha_1 = \alpha_2 = .05$, and $\omega_1 = .5$, $\omega_2 = 0$.

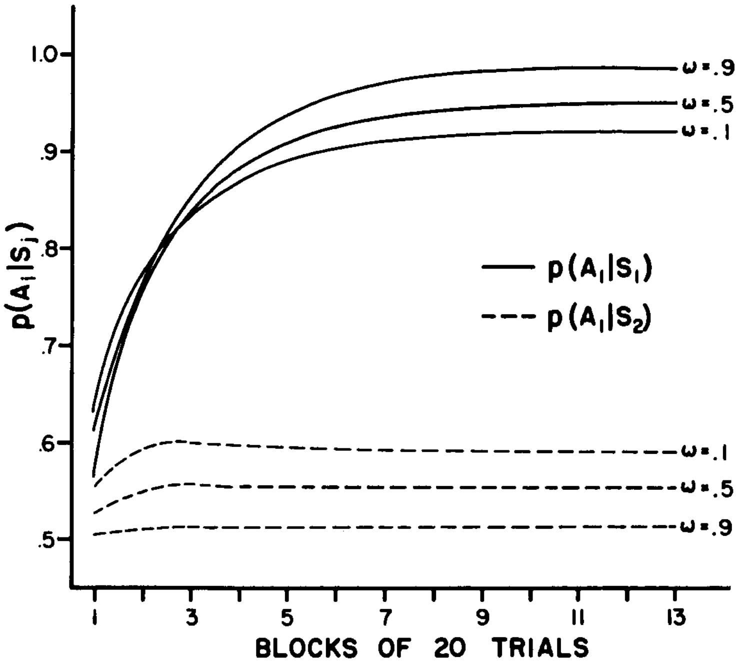


Figure 5. Theoretical functions averaged over 20-trial blocks for the Estes and Burke study.

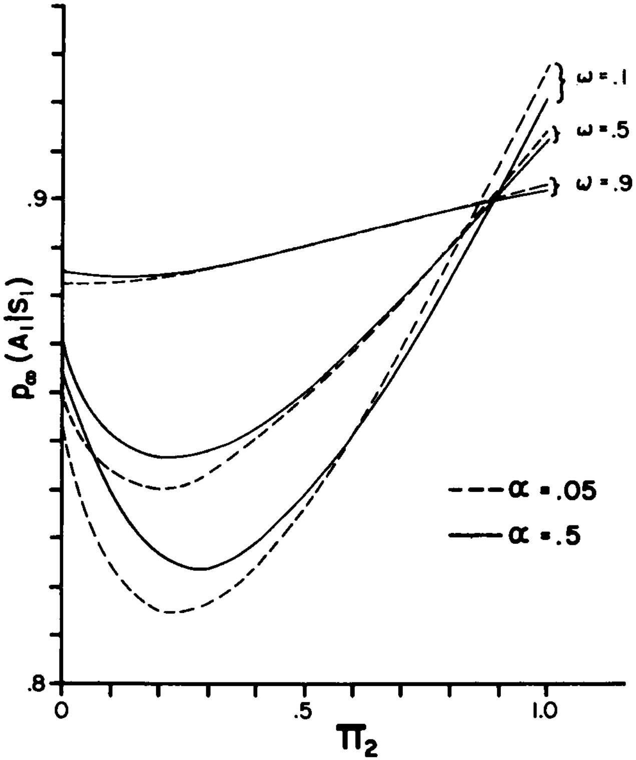


Figure 6. Predicted values of $p_\infty(A_1 \mid S_1)$ as a function of $\pi_2$; $\pi_1 = .9$ and $\beta = .5$.

Before the experiment was run, it was decided that the first block of ten trials and the last block of seventy trials would be analyzed separately. As predicted, over the last block of seventy trials the proportion of $A_1$ responses on $S_1$ trials was greater than the proportion of $A_2$'s on $S_2$ trials. A paired $t$ test indicated that the result was significant at the .05 level.

For the first block of ten trials, an additional restriction was placed on the random sequence of $S_1$ and $S_2$ events such that there were five $S_1$ trials and five $S_2$ trials. Over the first ten trials the proportion of $A_1$'s on $S_1$ trials was slightly less than the proportion of $A_2$'s on $S_2$ trials; however, this result did not approach statistical significance.

### *The Estes and Burke Study*

The case in which $\pi_1 = 1$, $\pi_2 = 1/2$, and $\beta = 1/2$ describes a discrimination task investigated by Estes and Burke [8]. There are several aspects to the study, but only the acquisition process for the "constant" group will be considered. Facing the subject was a circular array of twelve lights, and the onset of either the six lights on the left half of the panel or the six on the right half was designated an $S_1$ trial; the onset of the other six designated an $S_2$ trial. On each trial the subject made either an $A_1$ or an $A_2$ response and was then told which response was correct.

Figure 5 presents theoretical curves computed for $\alpha_1 = \alpha_2 = .05$, $u_i(1) = 1/16$, and $\omega_1 = \omega_2 = .1, .5$, and $.9$. The functions $p_n(A_1 \mid S_1)$ and $p_n(A_1 \mid S_2)$ for $\omega = .1$ provide a reasonably good fit of the observed data.

An inspection of the curves in Figure 5 illustrates some theoretical results for this special case, namely that the closer $p_\infty(A_1 \mid S_1)$ is to unity and the closer $p_\infty(A_1 \mid S_2)$ is to .5, the larger is the value of $\omega$.

### *Other Studies*

Two studies deal with the case where $\pi_1$ is fixed and $\pi_2$ varies over experimental groups. Popper and Atkinson [14] report a study involving five groups in which $\pi_1 = .85$, $\beta = .50$, and $\pi_2$ takes on the values .85, .70, .50, .30, and .15. A similar study is reported by Atkinson, Bogartz, and Turner [2] in which $\pi_1 = .9$ and $\pi_2$ is .9, .7, .5, .3, and .1 for the five groups, respectively. The findings of these studies are in agreement. Consequently, the discussion will be limited to the latter study.

The experimental procedure is similar to Estes and Burke [8]. Each trial begins with the onset of one of two lights ($S_1$ or $S_2$). The subject makes a response ($A_1$ or $A_2$) and is then told which response was correct ($E_1$ or $E_2$).

Figure 6 presents theoretical curves for $p_\infty(A_1 \mid S_1)$ for various values of $\omega = \omega_1 = \omega_2$ and $\alpha = \alpha_1 = \alpha_2$; $\pi_1$ and $\beta$ are fixed at .9 and .5, respectively. The abscissa represents different values of $\pi_2$ and the parameters are $\omega$ and $\alpha$. As $\alpha$ approaches 1 and also as $\omega$ approaches 1, $p_\infty(A_1 \mid S_1)$ approaches $\pi_1$ for all values of $\pi_2$. However, in general, the model predicts a convex relation between $p_\infty(A_1 \mid S_1)$ and $\pi_2$. This convex relation for the asymptotic probability of an $A_1$ response on $S_1$ trials was demonstrated in the Atkinson, Bogartz, and Turner study and also in the Popper and Atkinson study.

Figure 6 presents another interesting feature of Case II. Here we observe that when $\pi_1 = \pi_2 = .9$, the probability $p_\infty(A_1 \mid S_1)$ is generally greater than .9, with maximal differences between .9 and $p_\infty(A_1 \mid S_1)$ for small values of $\omega$ and $\alpha$. This result obtains in general for Case II. That is, if $\pi_1 = \pi_2 = \pi > 1/2$, then $p_\infty(A_1 \mid S_1)$ and $p_\infty(A_1 \mid S_2)$ are both greater than $\pi$ except when $\omega = 1$ or when $\alpha = 1$.

## 10. Discussion

An empirical evaluation of the theory is currently in progress. The research involves experimentation with both rats and human subjects. The variables being analyzed encompass various stimulus and reinforcement schedules and also include procedures designed to manipulate the values of $\omega_1$ and $\omega_2$. The research with rats is being conducted using two-bar Skinner boxes in which the bars are retractable, thereby permitting a discrete trial procedure. The equipment is completely automatic, and the animal's response data can be immediately transferred to I.B.M. cards for analysis.

Concurrently, various procedures for estimating parameters are being explored. In any application of the model it will be necessary to estimate at least two parameters. For example, if the stimuli $S_1$ and $S_2$ are comparable, then application of Case I would involve estimating $\omega$ and $\theta$. A maximum-likelihood method has been worked out for this case and is programmed at the Western Data Processing Center for the I.B.M. 709 computer. An individual subject's response record and sequence of experimental events are read into the computer, and values of $\omega$ and $\theta$ are computed that maximize the likelihood of the particular response protocol. Some preliminary work indicates that Case I will have very restricted applicability. Consequently, a similar procedure for Case II is being developed that permits the simultaneous estimation of $\omega$, $\alpha_1$, and $\alpha_2$. Other procedures for estimation, including a pseudo maximum-likelihood method suggested by Suppes, are also being investigated.

From a psychological standpoint there is one additional comment to be made about applications of the model. This involves a prediction, suggested by the work of Wyckoff [20], of the rate of change in perceptual and conditioning states. If the perceptual and conditioning processes are viewed as response mechanisms, then we would expect, in terms of a gradient of reinforcement [12], that the response temporally closest to the reinforcing event should be acquired most rapidly. Thus we might suspect that the rate of change in the conditioning states would be greater than in the perceptual states. For situations in which this analysis is correct, Case I is definitely not applicable. However, Case II may represent a good approximation when $\alpha_1 < \alpha_2$. More generally, such an analysis would suggest that $\gamma_1 < \gamma_2$ for Case III.

In conclusion, it appears that the model generates interesting predictions regarding both reinforcement schedules and similarity between discriminanda. No rigorous attempt has been made to test the theory for the special cases considered. Nevertheless, qualitatively it appears that the model accounts for some aspects of traditional types of discrimination learning and can be extended without modification to discrimination tasks involving more complex stimulus and reinforcement schedules.

## APPENDIX

Listed below are the transition probabilities for Case III; only non-zero terms are indicated. Interpret $p_{i,j}$ as the probability of being in state $j$ on trial $n+1$, given state $i$ on trial $n$.

$p_{1,1} = a + c + (b + d)\gamma_0$

$p_{1,4} = d\gamma_2$

$p_{1,7} = b\gamma_2$

$p_{1,9} = (b + d)\gamma_1$

$p_{1,12} = d\gamma_3$

$p_{1,15} = b\gamma_3$

$p_{2,1} = c\gamma_2$

$p_{2,2} = a + (b + c)\gamma_0 + d(\gamma_1 + \gamma_0)$

$p_{2,4} = d(\gamma_3 + \gamma_2)$

$p_{2,8} = p_{1,7}$

$p_{2,9} = c\gamma_3$

$p_{2,10} = (b + c)\gamma_1$

$p_{2,16} = p_{1,15}$

$p_{3,1} = (a + c)(\gamma_3 + \gamma_2)$

$p_{3,3} = (a + c)(\gamma_1 + \gamma_0) + (b + d)\gamma_0$

$p_{3,4} = p_{1,4}$

$p_{3,7} = p_{1,7}$

$p_{3,11} = p_{1,9}$

$p_{3,12} = p_{1,12}$

$p_{3,15} = p_{2,16}$

$p_{4,1} = p_{2,1}$

$p_{4,2} = a(\gamma_3 + \gamma_2)$

$p_{4,4} = a(\gamma_1 + \gamma_0) + (b + c)\gamma_0 + d$

$p_{4,8} = p_{1,7}$

$p_{4,9} = p_{2,9}$

$p_{4,12} = p_{2,10}$

$p_{4,16} = p_{2,16}$

$p_{5,1} = a\gamma_2$

$p_{5,5} = (a + d)\gamma_0 + b(\gamma_1 + \gamma_0) + c$

$p_{5,7} = b(\gamma_3 + \gamma_2)$

$p_{5,8} = p_{1,4}$

$p_{5,9} = a\gamma_3$

$p_{5,13} = (a + d)\gamma_1$

$p_{5,16} = p_{1,12}$

$p_{6,2} = p_{5,1}$

$p_{6,5} = p_{4,1}$

$p_{6,6} = (a + c)\gamma_0 + (b + d)(\gamma_1 + \gamma_0)$

$p_{6,8} = (b + d)(\gamma_3 + \gamma_2)$

$p_{6,10} = p_{5,9}$

$p_{6,13} = p_{4,9}$

$p_{6,14} = (a + c)\gamma_1$

$p_{7,1} = p_{5,1}$

$p_{7,5} = c(\gamma_3 + \gamma_2)$

$p_{7,7} = (a + d)\gamma_0 + b + c(\gamma_1 + \gamma_0)$

$p_{7,8} = p_{5,8}$

$p_{7,9} = p_{5,9}$

$p_{7,15} = p_{5,13}$

$p_{7,16} = p_{5,16}$

$p_{8,2} = p_{7,1}$

$p_{8,5} = p_{6,5}$

$p_{8,8} = (a + c)\gamma_0 + b + d$

$p_{8,10} = p_{6,10}$

$p_{8,13} = p_{6,13}$

$p_{8,16} = p_{6,14}$

$p_{9,1} = p_{3,11}$

$p_{9,4} = p_{5,16}$

$p_{9,7} = p_{1,15}$

$p_{9,9} = a + c + (b + d)\gamma_0$

$p_{9,12} = p_{5,8}$

$p_{9,15} = p_{1,7}$

$p_{10,1} = c\omega_2\gamma_3$

$p_{10,2} = [b + c\omega_2 + d(1 - \omega_2)]\gamma_1$

$p_{10,4} = d(1 - \omega_2)\gamma_3$

$p_{10,8} = p_{1,15}$

$p_{10,9} = c[\omega_2\gamma_2 + (1 - \omega_2)(\gamma_3 + \gamma_2)]$

$$\begin{aligned} p_{10,10} = a &+ b\gamma_0 \\ &+ c[\omega_2\gamma_0 + (1 - \omega_2)(\gamma_1 + \gamma_0)] \\ &+ d[\omega_2(\gamma_1 + \gamma_0) + (1 - \omega_2)\gamma_0] \end{aligned}$$

$p_{10,12} = d[\omega_2(\gamma_3 + \gamma_2) + (1 - \omega_2)\gamma_2]$

$p_{10,16} = p_{9,15}$

$p_{11,1} = [a(1 - \omega_1) + c(1 - \omega_2)]\gamma_3$

$$\begin{aligned} p_{1,13} = [&a(1 - \omega_1) + b\omega_1 + c(1 - \omega_2) \\ &+ d\omega_2]\gamma_1 \end{aligned}$$

$p_{11,4} = d\omega_2\gamma_3$

$p_{11,7} = b\omega_1\gamma_3$

$p_{11,9} = a[\omega_1(\gamma_3 + \gamma_2) + (1 - \omega_1)\gamma_2] + c[\omega_2(\gamma_3 + \gamma_2) + (1-\omega_2)\gamma_2]$

$p_{11,11} = a[\omega_1(\gamma_1 + \gamma_0) + (1 - \omega_1)\gamma_0] + b[\omega_1\gamma_0 + (1 - \omega_1)(\gamma_1 + \gamma_0)] + c[\omega_2(\gamma_1 + \gamma_0) + (1 - \omega_2)\gamma_0] + d[\omega_2\gamma_0 + (1 - \omega_2)(\gamma_1 + \gamma_0)]$

$p_{11,12} = d[\omega_2\gamma_2 + (1 - \omega_2)(\gamma_3 + \gamma_2)]$

$p_{11,15} = b[\omega_1\gamma_2 + (1 - \omega_1)(\gamma_3 + \gamma_2)]$

$p_{12,1} = p_{8,13}$

$p_{12,2} = a(1 - \omega_1)\gamma_3$

$p_{12,4} = [a(1 - \omega_1) + b\omega_1 + c]\gamma_1$

$p_{12,8} = p_{11,7}$

$p_{12,9} = p_{4,1}$

$p_{12,10} = a[\omega_1(\gamma_3 + \gamma_2) + (1 - \omega_1)\gamma_2]$

$p_{12,12} = a[(\gamma_1 + \gamma_0)\omega_1 + (1 - \omega_1)\gamma_0] + b[\omega_1\gamma_0 + (1 - \omega_1)(\gamma_1 + \gamma_0)] + c\gamma_0 + d$

$p_{12,16} = p_{11,15}$

$p_{13,1} = a\omega_1\gamma_3$

$p_{13,5} = [a\omega_1 + b(1 - \omega_1) + d]\gamma_1$

$p_{13,7} = b(1 - \omega_1)\gamma_3$

$p_{13,8} = d\gamma_3$

$p_{13,9} = a[\omega_1\gamma_2 + (1 - \omega_1)(\gamma_3 + \gamma_2)]$

$p_{13,13} = a[\omega_1\gamma_0 + (1 - \omega_1)(\gamma_1 + \gamma_0)] + b[\omega_1(\gamma_1 + \gamma_0) + (1 - \omega_1)\gamma_0] + c + d\gamma_0$

$p_{13,15} = b[\omega_1(\gamma_3 + \gamma_2) + (1 - \omega_1)\gamma_2]$

$p_{13,16} = p_{5,8}$

$p_{14,2} = p_{13,1}$

$p_{14,5} = c\omega_2\gamma_3$

$p_{14,6} = [a\omega_1 + b(1 - \omega_1) + c\omega_2 + d(1 - \omega_2)]\gamma_1$

$p_{14,8} = [b(1 - \omega_1) + d(1 - \omega_2)]\gamma_3$

$p_{14,10} = p_{13,9}$

$p_{14,13} = c[\omega_2\gamma_2 + (1 - \omega_2)(\gamma_3 + \gamma_2)]$

$p_{14,14} = a[\omega_1\gamma_0 + (1 - \omega_1)(\gamma_1 + \gamma_0)] + b[\omega_1(\gamma_1 + \gamma_0) + (1 - \omega_1)\gamma_0] + c[\omega_2\gamma_0 + (1 - \omega_2)(\gamma_1 + \gamma_0)] + d[\omega_2(\gamma_1 + \gamma_0) + (1 - \omega_2)\gamma_0]$

$p_{14,16} = b[\omega_1(\gamma_3 + \gamma_2) + (1 - \omega_1)\gamma_2] + d[\omega_2(\gamma_3 + \gamma_2) + (1 - \omega_2)\gamma_2]$

$p_{15,1} = p_{5,9}$

$p_{15,5} = c(1 - \omega_2)\gamma_3$

$p_{15,7} = [a + c(1 - \omega_2) + d\omega_2]\gamma_1$

$p_{15,8} = d\omega_2\gamma_3$

$p_{15,9} = p_{5,1}$

$p_{15,13} = c[\omega_2(\gamma_3 + \gamma_2) + (1 - \omega_2)\gamma_2]$

$p_{15,15} = a\gamma_0 + b. + c[\omega_2(\gamma_1 + \gamma_0) + (1 - \omega_2)\gamma_0] + d[\omega_2\gamma_0 + (1 - \omega_2)(\gamma_1 + \gamma_0)]$

$p_{15,16} = p_{11,12}$

$p_{16,2} = p_{5,9}$

$p_{16,5} = p_{2,9}$

$p_{16,8} = (a + c)\gamma_1$

$p_{16,10} = p_{5,1}$

$p_{16,13} = p_{6,5}$

$p_{16,16} = (a + c)\gamma_0 + b + d$

## REFERENCES


[1] ATKINSON, R. C. "A Markov Model for Discrimination Learning," *Psychometrika*, **23** (1958), 309-22.

[2] ATKINSON, R. C., W. H. BOGARTZ, and R. N. TURNER. "Supplementary Report: Discrimination Learning with Probabilistic Reinforcement Schedules," *Journal of Experimental Psychology*, **57** (1959), 349-50.

[3] BURKE, C. J., and W. K. ESTES. "A Component Model for Stimulus Variables in Discrimination Learning," *Psychometrika*, **22** (1957), 133-45.

[4] BUSH, R. R., and F. MOSTELLER. "A Model for Stimulus Generalization and Discrimination," *Psychological Review*, **58** (1951), 413-23.

[5] ESTES, W. K. "Theory of Learning with Constant, Variable, or Contingent Probabilities of Reinforcement," *Psychometrika*, **22** (1957), 113-32.

[6] ESTES, W. K. "Of Models and Men," *The American Psychologist*, **12** (1957), 609-17.

[7] ESTES, W. K. "Component and Pattern Models with Markovian Interpretations," in R. Bush and W. K. Estes, eds., *Studies in Mathematical Learning Theory*, Stanford, Calif.: Stanford University Press, 1959.

[8] ESTES, W. K., and C. J. BURKE. "Application of a Statistical Model to Simple Discrimination Learning in Human Subjects," *Journal of Experimental Psychology*, **50** (1955), 81-88.

[9] ESTES, W. K., C. J. BURKE, R. C. ATKINSON, and J. P. FRANKMANN. "Probabilistic Discrimination Learning," *Journal of Experimental Psychology*, **54** (1957), 233-39.

[10] ESTES, W. K., and P. SUPPES. "Foundations of Linear Models," in R. Bush and W. K. Estes, eds., *Studies in Mathematical Learning Theory*, Stanford. Calif.: Stanford University Press, 1959.

[11] GREEN, E. J. "A Simplified Model for Stimulus Discrimination," *Psychological Review*, **65** (1958), 56-63.

[12] HULL, C. L. *Principles of Behavior*, New York: Appleton-Century-Crofts, 1943.

[13] LABERGE, D. L., and A. SMITH. "Selective Sampling in Discrimination Learning," *Journal of Experimental Psychology*, **54** (1957), 423-30.

[14] POPPER, J., and R. C. ATKINSON. "Discrimination Learning in a Verbal Conditioning Situation," *Journal of Experimental Psychology*, **56** (1958), 21-25.

[15] RESTLE, F. "A Theory of Discrimination Learning," *Psychological Review*, **62** (1955), 11-19.

[16] RESTLE, F. "Axioms of a Theory of Discrimination Learning," *Psychometrika*, **20** (1955), 201-8.

[17] RESTLE, F. "A Theory of Selective Learning with Probable Reinforcements," *Psychological Review*, **64** (1957), 182-91.

[18] RESTLE, F. "A Survey and Classification of Learning Models," in R. Bush and W. K. Estes, eds., *Studies in Mathematical Learning Theory*, Stanford, Calif.: Stanford University Press, 1959.

[19] SUPPES, P., and R. C. ATKINSON. "Markov Learning Models for Multiperson Situations, I. The Theory," Technical Report No. 21, Contract Nonr 225(17), Applied Mathematics and Statistics Laboratory, Stanford University, 1959.

[20] WYCKOFF, L. B., JR. "The Role of Observing Responses in Discrimination Behavior," *Psychological Review*, **59** (1952), 431-42.